Conformal invariance and action-at-a-distance electrodynamics

1974 J. Phys. A: Math. Nucl. Gen. 7 1817

(http://iopscience.iop.org/0301-0015/7/15/005)

J. Phys. A: Math., Nucl. Gen., Vol. 7, No. 15, 1974. Printed in Great Britain.

Conformal invariance and action-at-a-distance electrodynamics

L H Ryder†

Institut für theoretische Physik der Universität Karlsruhe, Karlsruhe, West Germany

Received 20 June 1973, in final form 3 June 1974

Abstract. The Fokker–Wheeler–Feynman theory of action-at-a-distance electrodynamics is not conformally invariant. It is shown how to modify it to restore conformal invariance, and it is proved that the modified theory yields Maxwell's equations and gives Dirac's expression for the radiation reaction.

1. Introduction

Very shortly after Einstein's first paper on special relativity, it was pointed out by Cunningham (1909) and Bateman (1910) that Maxwell's equations of electrodynamics are invariant not only under the ten parameter group of Lorentz transformations and translations in space–time, but under the fifteen parameter conformal group. (For a modern derivation of this result, see Bopp (1959).) Briefly put, the reason for this extended invariance is that the photon is massless, and in fact conformal invariance generally holds for theories involving massless particles (see, for example, Flato *et al* 1970). Because of this one would expect that any derivation of classical electrodynamics from a principle of least action should also start from an action which is conformally invariant.

One such derivation has been presented by Wheeler and Feynman (1945, 1949), in their theory of action-at-a-distance electrodynamics. They make use of an action which was originally written down by Fokker (1929), according to which particles interact by means of both the retarded *and advanced* Liénard–Wiechert potentials. It is this time-reversal symmetry which allows one to write down an action in the first place. What Wheeler and Feynman showed was that this action-at-a-distance electrodynamics need not violate causality, and is therefore worthy of more serious consideration than it might otherwise have attracted.

It is the purpose of this paper firstly to point out that the Fokker action of Wheeler and Feynman is not conformally invariant, and secondly, to propose a modification of the Fokker–Wheeler–Feynman action which is conformally invariant. It is shown that the electromagnetic field tensor in the modified theory satisfies Maxwell's equations and gives the same radiation reaction as that derived by Dirac (1938).

In § 2 we give a brief summary of conformal transformations, and in § 3 consider the transformation of fields and currents. In § 4 it is shown that the Fokker–Wheeler–Feynman action is not conformally invariant, and a modified action, having conformal

† On study leave from Physics Laboratories, University of Kent, Canterbury, UK.

symmetry, is written down. It is shown that the correspondingly modified electromagnetic field tensor satisfies Maxwell's equations. In § 5 the radiation reaction is derived, and § 6 concludes with some general remarks.

2. Conformal transformations

This section will be fairly brief; for more thorough reviews of conformal transformations, the reader is referred to Wess (1960) and Fulton *et al* (1962). Conformal transformations induce scale transformations in space–time by a factor $\lambda(x)$ which itself depends on the coordinates x^μ:

$$\mathrm{d}\bar{x}^\mu\, \mathrm{d}\bar{x}_\mu = \lambda(x)\, \mathrm{d}x^\mu\, \mathrm{d}x_\mu. \tag{1}$$

The light cone is left invariant, and for this reason, as was mentioned above, conformal transformations acquire considerable significance in theories of massless particles, like photons and gravitons. The easiest way of obtaining an expression for the most general conformal transformation is to consider the discrete transformation of inversion in the hypersphere

$$x'^\mu = \frac{x^\mu}{x^2}.$$

If we now perform the sequence inversion–translation–inversion

$$x'^\mu = \frac{x^\mu}{x^2}, \qquad x''^\mu = x'^\mu + c^\mu, \qquad \bar{x}^\mu = \frac{x''^\mu}{(x'')^2}$$

then we see that

$$\bar{x}^\mu = \frac{x^\mu + c^\mu x^2}{1 + 2c\,.\,x + c^2 x^2}. \tag{2}$$

This is the most general conformal transformation. It follows from (2) that

$$\bar{x}^\mu \bar{x}_\mu = (\bar{x})^2 = \frac{x^2}{1 + 2c\,.\,x + c^2 x^2} = \frac{x^2}{\sigma(x)} \tag{3}$$

where

$$\sigma(x) = 1 + 2c\,.\,x + c^2 x^2, \tag{4}$$

and that

$$(\bar{x} - \bar{y})^2 = \sigma^{-1}(x)\sigma^{-1}(y)(x - y)^2 \tag{5}$$

so that

$$(\mathrm{d}\bar{x})^2 = \sigma^{-2}(x)(\mathrm{d}x)^2 \tag{6}$$

as in (1), where we identify

$$\lambda(x) = \sigma^{-2}(x). \tag{7}$$

From (3), if $\sigma(x) = c^2[x + (c/c^2)]^2 < 0$, the sign of $\bar{x}^2$ is different from that of x^2, so a time-like vector becomes space-like and vice versa. Because this heralds a possible violation of causality, a common attitude to conformal invariance in physics is to believe that if it is relevant at all, then it is only *infinitesimal* transformations that are relevant. This is the attitude we propose to adopt.

Because the interval $(\mathrm{d}x)^2$ is not invariant, as seen in (1), the metric tensor does not behave as a true tensor under conformal transformations, but as a tensor density

$$\bar{g}^{\mu\nu}(\bar{x}) = \sigma^2(x)\frac{\partial\bar{x}^\mu}{\partial x^\rho}\frac{\partial\bar{x}^\nu}{\partial x^\sigma}g^{\rho\sigma}(x) \tag{8}$$

$$\bar{g}_{\mu\nu}(\bar{x}) = \sigma^{-2}(x)\frac{\partial x^\rho}{\partial\bar{x}^\mu}\frac{\partial x^\sigma}{\partial\bar{x}^\nu}g_{\rho\sigma}(x). \tag{9}$$

Since, by definition, the contravariant components of $\mathrm{d}x^\mu$ transform as

$$\mathrm{d}\bar{x}^\mu = \frac{\partial\bar{x}^\mu}{\partial x^\alpha}\mathrm{d}x^\alpha \tag{10}$$

then

$$\begin{aligned}(\mathrm{d}\bar{x})^2 &= \bar{g}_{\mu\nu}(\bar{x})\,\mathrm{d}\bar{x}^\mu\,\mathrm{d}\bar{x}^\nu\\ &= \sigma^{-2}(x)\frac{\partial x^\rho}{\partial\bar{x}^\mu}\frac{\partial x^\sigma}{\partial\bar{x}^\nu}\frac{\partial\bar{x}^\mu}{\partial x^\alpha}\frac{\partial\bar{x}^\nu}{\mathrm{d}x^\beta}g_{\rho\sigma}(x)\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta\\ &= \sigma^{-2}(x)\delta^\rho_\alpha\delta^\sigma_\beta g_{\rho\sigma}(x)\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = \sigma^{-2}(x)\,(\mathrm{d}x)^2\end{aligned} \tag{11}$$

as in (6).

The operator $\partial_\mu = \partial/\partial x^\mu$ transforms like a covariant vector

$$\bar{\partial}_\mu = \frac{\partial x^\rho}{\partial\bar{x}^\mu}\partial_\rho \tag{12}$$

while $\partial^\mu \equiv g^{\mu\nu}\partial_\nu$ transforms as a contravariant tensor density

$$\bar{\partial}^\mu = \sigma^2(x)\frac{\partial\bar{x}^\mu}{\partial x^\rho}\partial^\rho. \tag{13}$$

It is characteristic of conformal transformations that the covariant and contravariant components of tensors transform with different weight factors. This is because $g^{\mu\nu}$ is a tensor density.

Momentum p^μ transforms in the same way as ∂^μ, so we may deduce the transformation law for mass

$$\bar{m}^2 = \bar{p}^\mu\bar{p}_\mu = \sigma^2(x)\frac{\partial\bar{x}^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial\bar{x}^\mu}p^\rho p_\sigma = \sigma^2(x)p^\rho p_\rho = \sigma^2(x)m^2,$$

that is,

$$\bar{m} = \sigma(x)m = (1+2c\,.\,x+c^2x^2)m. \tag{14}$$

This is quite pathological behaviour. The mass of a particle changes, according to (14), under a conformal transformation, and, what is more, in a particular 'conformal frame' defined by c^μ, it changes along the particle's own world line. The physical meaning of this is, to say the least, obscure; the only thing we may say is that there is no difficulty if $m = 0$. We see again the special role of massless particles in conformal theories.

Finally, we require the transform of the Dirac delta function. If $\sigma(x)$, $\sigma(y) > 0$, it follows from (5) that

$$\delta((\bar{x}-\bar{y})^2) = \sigma(x)\sigma(y)\delta((x-y)^2). \tag{15}$$

3. Transformation of fields and currents

Since $d\bar{x}^\mu = (\partial\bar{x}^\mu/\partial x^\alpha)\, dx^\alpha$, equation (1) may be written

$$g_{\mu\nu}\frac{\partial\bar{x}^\mu}{\partial x^\alpha}\frac{\partial\bar{x}^\nu}{\partial x^\beta} = \lambda(x)g_{\alpha\beta} = \sigma^{-2}(x)g_{\alpha\beta} \tag{16}$$

where (7) has been used. Taking the determinant of (16) gives

$$\det\frac{\partial\bar{x}}{\partial x} = \sigma^{-4}(x) = \lambda^2(x). \tag{17}$$

It then follows from (16) and (17) that

$$\Lambda^\mu{}_\nu = \left|\det\frac{\partial\bar{x}}{\partial x}\right|^{-1/4}\frac{\partial\bar{x}^\mu}{\partial x^\nu}$$

is a Lorentz matrix

$$\Lambda^\mu{}_\nu\Lambda^\nu{}_\rho = \delta^\mu_\rho.$$

A field ψ, since it has well defined behaviour under Lorentz transformations, will then behave under a conformal transformation as (Isham *et al* 1970)

$$\psi(x) \to \psi'(\bar{x}) = \left|\det\frac{\partial\bar{x}}{\partial x}\right|^{l/4} D\left(\left|\det\frac{\partial\bar{x}}{\partial x}\right|^{-1/4}\frac{\partial\bar{x}^\mu}{\partial x^\nu}\right)\psi(x).$$

l is a Lorentz scalar, called the *weight*. As an example, a tensor field $B^\mu(x)$ of weight l, will transform as

$$\bar{B}^\mu(\bar{x}) = \left|\det\frac{\partial\bar{x}}{\partial x}\right|^{(l-1)/4}\frac{\partial\bar{x}^\mu}{\partial x^\nu}B^\nu(x) = \sigma^{1-l}(x)\frac{\partial\bar{x}^\mu}{\partial x^\nu}B^\nu(x). \tag{18}$$

It follows from (9) that $B_\mu(x) \equiv g_{\mu\nu}B^\nu(x)$ transforms as

$$\bar{B}_\mu(\bar{x}) = \sigma^{-1-l}(x)\frac{\partial x^\nu}{\partial\bar{x}^\mu}B_\nu(x), \tag{19}$$

in further demonstration of the fact that the covariant and contravariant components of a tensor (density) field of a given weight l, pick up different powers of σ under conformal transformations.

We assume that the electromagnetic four-vector potential $A^\mu(x)$ transforms like ∂^μ, in accordance with the usual prescription $\partial^\mu \to \partial^\mu - ieA^\mu$ for the introduction of electromagnetic interactions. Comparing (13) with (18) implies that $l = -1$:

$$\begin{aligned}\bar{A}^\mu(\bar{x}) &= \sigma^2(x)\frac{\partial\bar{x}^\mu}{\partial x^\nu}A^\nu(x)\\ \bar{A}_\mu(\bar{x}) &= \frac{\partial x^\nu}{\partial\bar{x}^\mu}A_\nu(x).\end{aligned} \tag{20}$$

An invariant action must have $l = 0$; this implies that the Lagrange density $\mathscr{L}$ must have $l = -4$. If the interaction Lagrangian is

$$\mathscr{L}_{\text{int}} = eA^\mu j_\mu$$

this implies that the current density j_μ must have $l = -3$, so

$$\bar{j}^\mu(\bar{x}) = \sigma^4(x)\frac{\partial \bar{x}^\mu}{\partial x^\nu} j^\nu(x)$$
$$\bar{j}_\mu(\bar{x}) = \sigma^2(x)\frac{\partial x^\nu}{\partial \bar{x}^\mu} j_\nu(x). \tag{21}$$

3.1. Maxwell's equations

The two homogeneous Maxwell's equations curl $\boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$ and div $\boldsymbol{B} = 0$ have the respective solutions $\boldsymbol{E} = -\text{grad}\, \phi - \partial \boldsymbol{A}/\partial t$ and $\boldsymbol{B} = \text{curl}\, \boldsymbol{A}$, or, in covariant notation

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu . \tag{22}$$

The inhomogeneous equations div $\boldsymbol{E} = \rho$ and $-(\partial \boldsymbol{E}/\partial t) + \text{curl}\, \boldsymbol{B} = \boldsymbol{j}$ are then

$$\partial^\mu F_{\mu\nu} = j_\nu . \tag{23}$$

Substitution of (22) into (23) gives

$$\Box^2 A^\nu - \partial^\nu \partial_\mu A^\mu = j^\nu . \tag{24}$$

In the Lorentz gauge

$$\partial_\mu A^\mu = 0 \tag{25}$$

(24) becomes

$$\Box^2 A^\nu = j^\nu \tag{26}$$

whose solution is the (advanced and/or retarded) Liénard–Wiechert potential.

Now it is straightforward to verify, using (20) and (21), that both sides of equation (23) behave in the same way under conformal transformations. However, equation (25) is not conformally invariant. By direct computation from (2)

$$\frac{\partial \bar{x}^\mu}{\partial x^\nu} = (1 + 2c\,.\,x + c^2x^2)^{-2}[\delta^\mu_\nu(1 + 2c\,.\,x + c^2x^2)$$
$$+ 2(c^\mu x_\nu - c_\nu x^\mu) + 4c^\mu x_\nu c\,.\,x - 2(c^\mu c_\nu x^2 + x^\mu x_\nu c^2)],$$

and using this, together with equations (4), (12) and (20) gives, to order c,

$$\partial_\mu A^\mu = 0 \Rightarrow \bar{\partial}_\mu \bar{A}^\mu = -4c_\nu A^\nu . \tag{27}$$

Thus (25) and (26) are not conformally invariant; conformal transformations and gauge transformations are not independent. It follows of course, that even in the absence of interactions, the equation $\Box^2 A^\mu = 0$ is not conformally invariant. The easiest way to see this is to note the highly irregular transformation law for the d'Alembertian

$$\bar{\Box}^2 = \sigma^2 \Box^2 + 8\sigma^2 c_\mu \partial^\mu$$

to order c.

4. Modification of the Fokker–Wheeler–Feynman action

In action-at-a-distance electrodynamics no reference is made to a field. Instead, particles interact directly (though the signals propagate with finite velocity), in such a way that the action

$$J = -\sum_x m_x c \int (\mathrm{d}x_\mu \, \mathrm{d}x^\mu)^{1/2} + \sum_{x<y} \frac{e_x e_y}{c} \int \int \delta((x-y)^2) \, \mathrm{d}x_\nu \, \mathrm{d}y^\nu \tag{28}$$

is an extremum. This is Fokker's action (Fokker 1929) and is used by Wheeler and Feynman (1949). In (28), the symbols x and y do service both for particle labels and for the distance in space–time along their world lines. Note that in the second, interaction, term in J, self-interactions are omitted. Thus the usual infinities associated with them do not occur. At the same time, however, the delta function

$$\delta((x-y)^2) = \delta(R^2 - c^2\tau^2) = \frac{1}{2R}[\delta(R - c\tau) + \delta(R + c\tau)]$$

where $R = |\boldsymbol{x} - \boldsymbol{y}|$, $\tau = x^0 - y^0$, shows the presence of both advanced and retarded effects. Wheeler and Feynman (1945) were able to show that the assumption that all radiation is eventually absorbed implies that: (*a*) there is no violation of causality; (*b*) an oscillating charged particle experiences a radiation reaction of exactly the type postulated by Dirac (1938).

Now let us turn to considerations of conformal invariance. Since conformal transformations leave the light cone invariant, c is invariant, and it follows immediately from equations (11) and (14) that the first term in J is invariant. In the second term, however,

$$\mathrm{d}x_\nu \, \mathrm{d}y^\nu = g_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}y^\nu \tag{29}$$

is not covariant. We have

$$\mathrm{d}\bar{x}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\rho} \mathrm{d}x^\rho, \qquad \mathrm{d}\bar{y}^\nu = \frac{\partial \bar{y}^\nu}{\partial y^\sigma} \mathrm{d}y^\sigma \tag{30}$$

so that $\mathrm{d}x^\mu$ behaves as a vector *at* x, and $\mathrm{d}y^\nu$ as a vector *at* y, but the contraction (29) is not a scalar, for what would we put for $\bar{g}_{\mu\nu}$? Do we evaluate it at x or at y? $g_{\mu\nu}$ needs to be replaced by a 'conformal metric tensor' $h_{\mu\nu}$ (introduced by Boulware *et al* 1970) which has the property that its μ index transforms as a vector at x, and its ν index as a vector at y. This requirement is satisfied by

$$h_{\mu\nu}(x, y) = -\frac{1}{2}(x-y)^2 \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \ln(x-y)^2 = g_{\mu\nu} - 2(x-y)^{-2}(x-y)_\mu (x-y)_\nu . \tag{31}$$

It follows from (5) and (12) that

$$\bar{h}_{\mu\nu}(\bar{x}, \bar{y}) = \sigma^{-1}(x)\sigma^{-1}(y) \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial y^\beta}{\partial \bar{y}^\nu} h_{\alpha\beta}(x, y). \tag{32}$$

It then follows from (15), (30) and (32) that $\mathrm{d}x^\mu \, \mathrm{d}y^\nu \, h_{\mu\nu}(x, y)\delta((x-y)^2)$ is invariant

$$\mathrm{d}\bar{x}^\mu \, \mathrm{d}\bar{y}^\nu \, \bar{h}_{\mu\nu}(\bar{x}, \bar{y})\delta((\bar{x} - \bar{y})^2) = \mathrm{d}x^\mu \, \mathrm{d}y^\nu \, h_{\mu\nu}(x, y)\delta((x-y)^2) \tag{33}$$

and hence that Fokker's action (28) is not conformally invariant, but that the modified action J^{C} is, where

$$J^{\mathrm{C}} = -\sum_x m_x c \int (\mathrm{d}x_\mu \, \mathrm{d}x^\mu)^{1/2} + \sum_{x<y} \frac{e_x e_y}{c} \int \int \delta((x-y)^2) h_{\mu\nu}(x, y) \, \mathrm{d}x^\mu \, \mathrm{d}y^\nu . \tag{34}$$

In order to discover the consequences of using this action, we need an expression for the effective vector potential. Wheeler and Feynman point out that if we define

$$A_\mu^{\mathrm{WF}}(x) =: e\int \delta((x-y)^2)\,\mathrm{d}y_\mu \tag{35}$$

as the effective vector potential induced at x by a particle of charge e moving along a world line y, then the condition that the action J in equation (28) is an extremum under variation of the world line of particle x results in the desired Lorentz equation of motion. $A_\mu^{\mathrm{WF}}(x)$ automatically obeys the Lorentz gauge condition

$$\partial^\mu A_\mu^{\mathrm{WF}}(x) = 0. \tag{36}$$

Now it follows from (15), (20) and (30) that the definition (35) is not conformally covariant, for the two sides of the equation have different transformation properties under conformal transformations. However, the condition that the Lorentz equation of motion be satisfied if the modified action J^{C} is an extremum under variation of the world line of x, is that the vector potential be given by

$$A_\mu^{\mathrm{C}}(x) =: e\int \delta((x-y)^2)h_{\nu\mu}(y,x)\,\mathrm{d}y^\nu = e\int \delta((x-y)^2)h_{\mu\nu}(x,y)\,\mathrm{d}y^\nu, \tag{37}$$

and from (20), (15), (32) and (30) it is seen that this is conformally covariant.

Wheeler and Feynman also define the current

$$j_\mu(x) = 4\pi e\int \delta^4(x-y)\,\mathrm{d}y_\mu. \tag{38}$$

Because of the identity

$$\Box^2\delta((x-y)^2) = 4\pi\delta^4(x-y) \tag{39}$$

it follows that (in the Lorentz gauge) Maxwell's equations (26) are satisfied when A_μ is given by (35) and j_μ by (38). Since I have found no proof of (39) in the literature, I give one in an appendix to this paper.

Is the definition (38) conformally covariant? From (9) and (10) it follows that

$$\mathrm{d}\bar{y}_\mu = \sigma^{-2}(y)\frac{\partial y^\nu}{\partial \bar{y}^\mu}\,\mathrm{d}y_\nu. \tag{40}$$

Since the four-dimensional delta function transforms by multiplication with the Jacobian, which is given by (17), we have

$$\delta^4(\bar{x}-\bar{y}) = \sigma^4(x-y)\delta^4(x-y). \tag{41}$$

It then follows that

$$\int \delta^4(\bar{x}-\bar{y})\,\mathrm{d}\bar{y}_\mu = \sigma^2(x)\frac{\partial x^\nu}{\partial \bar{x}^\mu}\int \delta^4(x-y)\,\mathrm{d}y_\nu$$

and hence that the right-hand side of (38) has the transformation properties required of a current, given in (21). So (38) is conformally covariant.

4.1. Maxwell's equations

Now that we have a conformally covariant vector potential given by (37) and current given by (38), we must show that they satisfy Maxwell's equations. We begin by using (31) to write

$$\begin{aligned} A^{\mathrm{C}}_\mu(x) &= e\int \delta((x-y)^2)h_{\mu\nu}(x,y)\dot{y}^\nu\,\mathrm{d}s \\ &= e\int \delta((x-y)^2)\dot{y}_\mu\,\mathrm{d}s - 2e\int \delta((x-y)^2)\frac{(x-y)_\mu(x-y)_\nu}{(x-y)^2}\dot{y}^\nu\,\mathrm{d}s \\ &=: A^{\mathrm{WF}}_\mu(x) - A^0_\mu(x), \end{aligned} \tag{42}$$

where the dot represents differentiation with respect to s, the world-line parameter of particle y. A^{WF}_μ is the Wheeler–Feynman vector potential given by (35). We know that it satisfies Maxwell's equations, so, since they are linear, we now have to show that A^0_μ also does. We need the identities

$$\begin{aligned} \frac{\partial}{\partial x^\mu}\delta((x-y)^2) &= 2(x-y)_\mu\delta'((x-y)^2) \\ &= -\frac{(x-y)_\mu}{\dot{y}^\lambda(x-y)_\lambda}\frac{\mathrm{d}}{\mathrm{d}s}\delta((x-y)^2) = -2\delta((x-y)^2)\frac{(x-y)_\mu}{(x-y)^2} \end{aligned} \tag{43}$$

where the prime represents differentiation with respect to the argument of the delta function, and the last equality follows from the well known identity $\delta'(x) = -\delta(x)/x$. We then have (where $\partial_\nu \equiv \partial/\partial x^\nu$)

$$\begin{aligned} \partial_\nu A^0_\mu(x) = 2e\int\Bigg(&-\frac{\mathrm{d}}{\mathrm{d}s}\delta((x-y)^2)\frac{(x-y)_\mu(x-y)_\nu}{(x-y)^2} + \delta((x-y)^2)\frac{g_{\mu\nu}(x-y)_\lambda\dot{y}^\lambda}{(x-y)^2} \\ &+ \delta((x-y)^2)\frac{(x-y)_\mu\dot{y}_\nu}{(x-y)^2} - 2\delta((x-y)^2)\frac{(x-y)_\mu(x-y)_\nu(x-y)_\lambda\dot{y}^\lambda}{(x-y)^4}\Bigg)\,\mathrm{d}s. \end{aligned} \tag{44}$$

Since the first two terms and the last one in (44) are symmetric in μ and ν, they do not contribute to $F^0_{\mu\nu} \equiv \partial_\mu A^0_\nu - \partial_\nu A^0_\mu$ which is therefore given by

$$F^0_{\mu\nu} = 2e\int\frac{\delta((x-y)^2)}{(x-y)^2}[(x-y)_\nu\dot{y}_\mu - (x-y)_\mu\dot{y}_\nu]\,\mathrm{d}s. \tag{45}$$

We note that $F^0_{\mu\nu} \neq 0$, so $F^{\mathrm{C}}_{\mu\nu} \neq F^{\mathrm{WF}}_{\mu\nu}$. We shall now show, however, that

$$\partial^\mu F^0_{\mu\nu} = 0 \tag{46}$$

so that

$$\partial^\mu F^{\mathrm{C}}_{\mu\nu} = \partial^\mu F^{\mathrm{WF}}_{\mu\nu} = j_\nu. \tag{47}$$

These are Maxwell's inhomogeneous equations (23). Making further use of (43), we have

$$\begin{aligned} \partial^\mu F^0_{\mu\nu} &= 2e\int\frac{\delta((x-y)^2)}{(x-y)^2}\left(\dot{y}_\nu - 4\frac{(x-y)_\nu(x-y)_\lambda\dot{y}^\lambda}{(x-y)^2}\right)\mathrm{d}s \\ &= -2e\int\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\delta((x-y)^2)}{(x-y)^2}(x-y)_\nu\right)\mathrm{d}s = 0. \end{aligned} \tag{48}$$

Maxwell's equations are therefore satisfied. However, since $F^{\mathrm{C}}_{\mu\nu} \neq F^{\mathrm{WF}}_{\mu\nu}$ we still have to check by explicit calculation that the two field tensors give the same radiation reaction, which, as Feynman and Wheeler (1945) showed, is the same as Dirac's. We do this in the next section. Before concluding this section, however, it is of interest to note that A_μ^{C} also obeys the Lorentz condition (25), as may be seen from (44). This does not contradict the conformal invariance of the theory, for conformal transformations and gauge transformations are not independent. It simply appears that action-at-a-distance theories select a particular gauge for A_μ, and it is neither here nor there that it happens to be the Lorentz gauge. What matters is that A_μ^{C}, as defined, is conformally covariant and satisfies Maxwell's equations.

5. Radiation reaction

First of all let us calculate $F_{\mu\nu}^{\mathrm{WF}}$. From (42) and (43) we have, by straightforward manipulation

$$F_{\mu\nu}^{\mathrm{WF}} = e\int \delta((x-y)^2)\Bigg(\frac{(x-y)_\mu \ddot{y}_\nu - (x-y)_\nu \ddot{y}_\mu}{(x-y)_\lambda \dot{y}^\lambda} - \frac{[\dot{y}_\lambda \dot{y}^\lambda + (x-y)_\lambda \ddot{y}^\lambda][(x-y)_\mu \dot{y}_\nu - (x-y)_\nu \dot{y}_\mu]}{[(x-y)_\lambda \dot{y}^\lambda]^2}\Bigg)\,\mathrm{d}s. \tag{49}$$

By using the formula (see Dirac 1938)

$$\int f(s)\delta(g(s))\,\mathrm{d}s = \frac{f(s)}{\dot{g}(s)}\bigg|_{g(s)=0} \tag{50}$$

equation (49) may be written

$$F_{\mu\nu}^{\mathrm{WF}} = \frac{e}{[(x-y)_\lambda \dot{y}^\lambda]^2}[(x-y)_\mu \ddot{y}_\nu - (x-y)_\nu \ddot{y}_\mu]|_{(x-y)^2=0} + \frac{e[(x-y)_\lambda \ddot{y}^\lambda + \dot{y}_\lambda \dot{y}^\lambda]}{[(x-y)_\lambda \dot{y}^\lambda]^3}[(x-y)_\mu \dot{y}_\nu - (x-y)_\nu \dot{y}_\mu]|_{(x-y)^2=0}. \tag{51}$$

By identifying s with proper time, we have $\dot{y}_\lambda \dot{y}^\lambda = 1$, and so the term with this coefficient in (51) varies as $(x-y)^{-2}$, whereas all the other terms vary as $(x-y)^{-1}$. This term may therefore be neglected. Wheeler and Feynman (1945) in the section 'Radiative reaction: derivation III' showed that the resulting expression gives, for the field of radiative reaction at the source

$$F_{\mu\nu,\mathrm{rad}} = \frac{2e}{3}(\dot{y}_\mu \dddot{y}_\nu - \dddot{y}_\mu \dot{y}_\nu) \tag{52}$$

in agreement with that given by Dirac.

In the conformally covariant theory, we have

$$F_{\mu\nu}^{\mathrm{C}} = F_{\mu\nu}^{\mathrm{WF}} - F_{\mu\nu}^{0} \tag{53}$$

where, from (45) and (50)

$$F_{\mu\nu}^{0} = \frac{2e}{(x-y)^2}\left(\frac{(x-y)_\nu \dot{y}_\mu - (x-y)_\mu \dot{y}_\nu}{(x-y)_\lambda \dot{y}^\lambda}\right)\bigg|_{(x-y)^2=0}. \tag{54}$$

We see immediately that $F_{\mu\nu}^{0}$ varies with separation between source and field points as $(x-y)^{-2}$, and may therefore be neglected in comparison with the dominant terms in $F_{\mu\nu}^{\mathrm{WF}}$. The resulting radiation reaction is then clearly given by Dirac's expression (52).

6. Concluding remarks

We have shown that it is possible to modify the Fokker–Wheeler–Feynman action to an explicitly conformally invariant form. This new action satisfies Maxwell's equations and gives the correct value for the radiation reaction of Dirac.

It may be of interest to note that Ramond (1973) has recently drawn attention to some parallels between action-at-a-distance theories and dual resonance models. By showing that action-at-a-distance electrodynamics can be made conformally invariant, it may be that we have uncovered another possible connection between the two theories, since dual resonance models have conformal invariance. However, it is important to note that our conformal transformations are in four-dimensional space–time, whereas dual models have conformal symmetry in two dimensions.

Acknowledgments

I wish to thank Dr J C Taylor for a helpful correspondence, Professor R Penrose for an interesting conversation, and Drs P H Dondi, W A B Evans and J McEwan, and Professors G Rickayzen and J Wess for helpful remarks. I also wish to thank Professor Wess for his kind hospitality at Karlsruhe, where this work was completed, and Karlsruhe University and the Deutsche Akademische Austauschdienst for financial support.

Appendix

We are to prove that

$$\Box^2\delta(x^2) = 4\pi\delta^4(x) \tag{A.1}$$

where $x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$, and $\Box^2 = \partial^\mu\partial_\mu = (\partial/\partial x_0^2) - \nabla^2$. We shall use, throughout this appendix, the notation $|\boldsymbol{x}| = r$. We first write

$$\delta(x^2) = \delta(x_0^2 - r^2) = \frac{1}{2r}[\delta(x^0 - r) + \delta(x^0 + r)]. \tag{A.2}$$

Using the identity

$$\delta'(x) = -\frac{\delta(x)}{x}$$

it is easy to see that

$$\frac{\partial^2}{\partial x_0^2}\delta(x^2) = \frac{1}{r}\left(\frac{\delta(x^0 - r)}{(x^0 - r)^2} + \frac{\delta(x^0 + r)}{(x^0 + r)^2}\right) \tag{A.3}$$

and that

$$\frac{\partial}{\partial x_i}[\delta(x^0 - r) + \delta(x^0 + r)] = \left(\frac{\delta(x^0 - r)}{x^0 - r} - \frac{\delta(x^0 + r)}{x^0 + r}\right)\frac{x_i}{r} \tag{A.4}$$

where $i = 1, 2, 3$, so that

$$\frac{\partial^2}{\partial x_i^2}[\delta(x^0-r)+\delta(x^0+r)]$$
$$= 2\left(\frac{\delta(x^0-r)}{(x^0-r)^2}+\frac{\delta(x^0+r)}{(x^0+r)^2}\right)\frac{x_i^2}{r^2}+\left(\frac{\delta(x^0-r)}{x^0-r}-\frac{\delta(x^0+r)}{x^0+r}\right)\left(\frac{1}{r}-\frac{x_i^2}{r^3}\right) \tag{A.5}$$

where i is not summed over, and

$$\nabla^2[\delta(x^0-r)+\delta(x^0+r)]$$
$$= 2\left(\frac{\delta(x^0-r)}{(x^0-r)^2}+\frac{\delta(x^0+r)}{(x^0+r)^2}\right)+\frac{2}{r}\left(\frac{\delta(x^0-r)}{x^0-r}-\frac{\delta(x^0+r)}{x^0+r}\right). \tag{A.6}$$

Now we have

$$\begin{aligned}\nabla^2\delta(x^2) = {} & \nabla^2\left(\frac{1}{2r}\right)[\delta(x^0-r)+\delta(x^0+r)] \quad \text{I}\\ & +\nabla\left(\frac{1}{r}\right).\nabla[\delta(x^0-r)+\delta(x^0+r)] \quad \text{II}\\ & +\frac{1}{2r}\nabla^2[\delta(x^0-r)+\delta(x^0+r)]. \quad \text{III}\end{aligned} \tag{A.7}$$

By using the identity

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(x)$$

we see that the first term in (A.7) is

$$\text{I} = -2\pi\delta^3(x)[\delta(x^0+r)+\delta(x^0-r)] = -4\pi\delta^4(x). \tag{A.8}$$

Similarly

$$\text{II} = -\frac{1}{r^2}\left(\frac{\delta(x^0-r)}{x^0-r}-\frac{\delta(x^0+r)}{x^0+r}\right) \tag{A.9}$$

and

$$\text{III} = \frac{1}{r}\left(\frac{\delta(x^0-r)}{(x^0-r)^2}+\frac{\delta(x^0+r)}{(x^0+r)^2}\right)+\frac{1}{r^2}\left(\frac{\delta(x^0-r)}{x^0-r}-\frac{\delta(x^0+r)}{x^0+r}\right). \tag{A.10}$$

(A.1) then follows from (A.3) and (A.7)–(A.10).

References

Bateman H 1910 *Proc. Lond. Math. Soc.* **8** 223
Bopp F 1959 *Ann. Phys. Lpz.* **4** 96–102
Boulware D G, Brown L S and Peccei R D 1970 *Phys. Rev.* D **2** 293–8
Cunningham E 1909 *Proc. Lond. Math. Soc.* **8** 77
Dirac P A M 1938 *Proc. R. Soc.* A **167** 148–69
Flato M, Simon J and Sternheimer D 1970 *Ann. Phys., NY* **61** 78–97

Fokker A D 1929 *Z. Phys.* **58** 386–93
Fulton T, Rohrlich F and Witten L 1962 *Rev. Mod. Phys.* **34** 442–57
Isham C, Salam A and Strathdee J 1970 *Phys. Rev.* D **2** 685–90
Ramond P 1973 *Phys. Rev.* D **7** 449–58
Wess J 1960 *Nuovo Cim.* **18** 1086–107
Wheeler J A and Feynman R P 1945 *Rev. Mod. Phys.* **17** 157–81
——1949 *Rev. Mod. Phys.* **21** 425–33